# A FIGURATRIX FOR DOUBLE INTEGRALS*

BY

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In a previous paper† the writer has called attention to a number of the interesting properties of the figuratrix in the calculus of variations. It is the purpose of the present paper to define a figuratrix for double integrals and to develop some of its important characteristics.

1. **A new form of the problem of double integrals.** The theory of extrema of double integrals has been treated for integrals of the form‡

$$\int\int f(x, y, z, z_x, z_y)dxdy, \tag{1}$$

in which $z$ and its partial derivatives are functions of $x$ and $y$, and also for integrals of the form§

$$\int\int F(x, y, z, x_u, y_u, z_u, x_v, y_v, z_v)dudv, \tag{2}$$

in which $x, y, z$ are the coördinates of a surface

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v). \tag{3}$$

Radon|| has developed the theory for a third form of integral,

$$\int\int F(x, y, z, A, B, C)dudv, \tag{4}$$

where the last three arguments of $F$ are the usual determinants of surface theory,

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† *The figuratrix in the calculus of variations*, these Transactions, vol. 28 (1926), pp. 640–653. This paper will be referred to as Rider, Transactions.

‡ See, for example, Bolza, *Vorlesungen über Variationsrechnung*, chapter 13.

§ See Kobb, *Sur les maxima et les minima des intégrales doubles*, Acta Mathematica, vol. 16 (1892–93), pp. 65–140, vol. 17 (1893), pp. 321–43; and Kneser, *Lehrbuch der Variationsrechnung*, 1900, pp. 263–306.

|| *Über einige Fragen betreffend die Theorie der Maxima und Minima mehrfachen Integrale*, Monatshefte für Mathematik und Physik, vol. 22 (1911), pp. 53–63. Radon considers a multiple integral, of which (4) is a special case. His notation is slightly different.

$$A = \begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix}, \quad B = \begin{vmatrix} z_u & x_u \\ z_v & x_v \end{vmatrix}, \quad C = \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix}. \tag{5}$$

It is found convenient in defining a figuratrix for double integrals to employ an integral of the form*

$$\int\int f(x, y, z, \tau, \sigma) H du dv, \tag{6}$$

where, in the usual notation of the theory of surfaces,

$$H = (EG - F^2)^{1/2} = (A^2 + B^2 + C^2)^{1/2}. \tag{7}$$

The quantities $\tau$ and $\sigma$ are angles defined by the equations

$$A = H \cos \tau \cos \sigma, \quad B = H \sin \tau \cos \sigma, \quad C = H \sin \sigma, \tag{8}$$

or

$$\tau = \arctan (B/A), \quad \sigma = \arcsin (C/H). \tag{9}$$

The geometric meaning of $\tau$ and $\sigma$ will be evident upon inspection of the accompanying figure, in which $PN$ is normal to the surface (3) at the point $P$, and $Pp$, $Pq$, $Pr$ are parallel to the coördinate axes $Ox$, $Oy$, $Oz$ respectively.

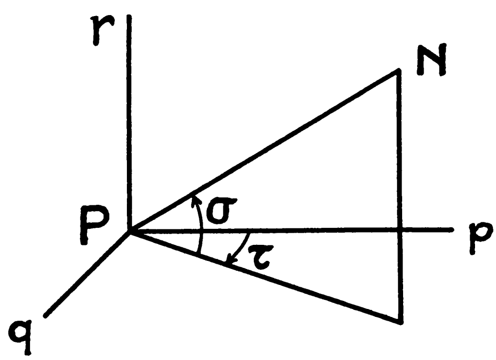


Fig. 1

The form (6) permits the use of the parametric representation of surfaces and avoids the homogeneity conditions required for integrals of form (2)† or form (4).‡ Moreover, it can be shown that any integral of

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* Cf. Rider, *The space problem of the calculus of variations in terms of angle*, American Journal of Mathematics, vol. 39 (1917), pp. 241–56.

† See Kobb, loc. cit., p. 68.

‡ See Radon, loc. cit., p. 55.

the form (1), or any integral of the forms (2) or (4) satisfying the required homogeneity condition, can be reduced to the form (6).

2. **The Euler-Lagrange equations.** In order to develop the equations which must be satisfied by surfaces that minimize or maximize the integral (6), let us assume that the surface which affords the least or the greatest value is given by equations (3). If we take a neighboring surface

$$\bar{x} = x + \epsilon\xi, \qquad \bar{y} = y + \epsilon\eta, \qquad \bar{z} = z + \epsilon\zeta \tag{10}$$

and integrate the function $fH$ over a region $R$ of the $uv$-plane, we have

$$I(\epsilon) = \int_R\int f(\bar{x}, \bar{y}, \bar{z}, \bar{\tau}, \bar{\sigma})\bar{H}dudv, \tag{11}$$

where a bar over a variable indicates that its values on the surface (10) is to be used. For example,

$$\bar{\tau} = \arctan\frac{\bar{B}}{\bar{A}}, \quad \bar{A} = \begin{vmatrix} \bar{y}_u & \bar{z}_u \\ \bar{y}_v & \bar{z}_v \end{vmatrix}, \quad \bar{B} = \begin{vmatrix} \bar{z}_u & \bar{x}_u \\ \bar{z}_v & \bar{x}_v \end{vmatrix}.$$

Obviously $dI/d\epsilon$ must be equal to zero for $\epsilon=0$ if $I$ is to be either a maximum or a minimum. We find that

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = \int_R\int\left[f\left.\frac{\partial\bar{H}}{\partial\epsilon}\right|_{\epsilon=0} + \left(f_x\xi + f_y\eta + f_z\zeta + f_\tau\left.\frac{\partial\bar{\tau}}{\partial\epsilon}\right|_{\epsilon=0}\right.\right. \tag{12}$$
$$\left.\left. + f_\sigma\left.\frac{\partial\bar{\sigma}}{\partial\epsilon}\right|_{\epsilon=0}\right)H\right]dudv.$$

By differentiating the equation $\bar{H}^2=\bar{A}^2+\bar{B}^2+\bar{C}^2$, we find that

$$\begin{aligned} \bar{H}\frac{\partial\bar{H}}{\partial\epsilon} &= \bar{A}\frac{\partial\bar{A}}{\partial\epsilon} + \bar{B}\frac{\partial\bar{B}}{\partial\epsilon} + \bar{C}\frac{\partial\bar{C}}{\partial\epsilon} \\ &= \bar{H}\left(\cos\bar{\tau}\cos\bar{\sigma}\frac{\partial\bar{A}}{\partial\epsilon} + \sin\bar{\tau}\cos\bar{\sigma}\frac{\partial\bar{B}}{\partial\epsilon} + \sin\bar{\sigma}\frac{\partial\bar{C}}{\partial\epsilon}\right). \end{aligned} \tag{13}$$

But

$$\left.\frac{\partial\bar{A}}{\partial\epsilon}\right|_{\epsilon=0} = \begin{vmatrix} \eta_u & z_u \\ \eta_v & z_v \end{vmatrix} + \begin{vmatrix} y_u & \zeta_u \\ y_v & \zeta_v \end{vmatrix},$$

and the values of $\partial\bar{B}/\partial\epsilon$ and $\partial\bar{C}/\partial\epsilon$ for $\epsilon=0$ can be obtained by cyclic permutation of $x$, $y$, $z$ and $\xi$, $\eta$, $\zeta$. Hence

$$\left.\frac{\partial H}{\partial \epsilon}\right|_{\epsilon=0} = (y_v \sin\sigma - z_v \sin\tau\cos\sigma)\xi_u + (z_v \cos\tau\cos\sigma - x_v \sin\sigma)\eta_u$$

$$(14) \qquad + (x_v \sin\tau\cos\sigma - y_v \cos\tau\cos\sigma)\zeta_u$$

$$- (y_u \sin\sigma - z_u \sin\tau\cos\sigma)\xi_v - (z_u \cos\tau\cos\sigma - x_u \sin\sigma)\eta_v$$

$$- (x_u \sin\tau\cos\sigma - y_u \cos\tau\cos\sigma)\zeta_v.$$

Now we can readily find that

$$(15) \qquad \left.\frac{\partial\bar{\tau}}{\partial \epsilon}\right|_{\epsilon=0} = \frac{1}{H}\Bigg[ - z_v\frac{\cos\tau}{\cos\sigma}\xi_u + z_v\frac{\sin\tau}{\cos\sigma}\eta_u + \left(x_v\frac{\cos\tau}{\cos\sigma} - y_v\frac{\sin\tau}{\cos\sigma}\right)\zeta_u + z_u\frac{\cos\tau}{\cos\sigma}\xi_v - z_u\frac{\sin\tau}{\cos\sigma}\eta_v - \left(x_v\frac{\cos\tau}{\cos\sigma} - y_u\frac{\sin\tau}{\cos\sigma}\right)\zeta_v\Bigg],$$

$$\left.\frac{\partial\bar{\sigma}}{\partial \epsilon}\right|_{\epsilon=0} = \frac{1}{H}[(y_v\cos\sigma + z_v\sin\tau\sin\sigma)\xi_u - (x_v\cos\sigma + z_v\cos\tau\sin\sigma)\eta_u$$

$$(16) \qquad - (x_v\sin\tau\sin\sigma - y_v\cos\tau\sin\sigma)\zeta_u - (y_u\cos\sigma + z_u\sin\tau\sin\sigma)\xi_v$$

$$+ (x_u\cos\sigma + z_u\cos\tau\sin\sigma)\eta_v + (x_u\sin\tau\sin\sigma$$

$$- y_u\cos\tau\sin\sigma)\zeta_v].$$

Substituting (14), (15), (16) in (12), and rearranging, we find that

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = \int_R\int [(f_x\xi + f_y\eta + f_z\zeta)H$$

$$(17) \qquad + (y_v r - z_v q)\xi_u + (z_v p - x_v r)\eta_u + (x_v q - y_v p)\zeta_u$$

$$- (y_u r - z_u q)\xi_v - (z_u p - x_u r)\eta_v - (x_u q - y_u p)\xi_v]dudv,$$

where

$$p = f\cos\tau\cos\sigma - f_\tau\sin\tau/\cos\sigma - f_\sigma\cos\tau\sin\sigma,$$

$$(18) \qquad q = f\sin\tau\cos\sigma + f_\tau\cos\tau/\cos\sigma - f_\sigma\sin\tau\sin\sigma,$$

$$r = f\sin\sigma + f_\sigma\cos\sigma.$$

With the usual assumptions regarding the continuity of $f$ and its derivatives, and with the further assumption that the functions $\xi$, $\eta$, $\zeta$ vanish on the contour of the region $R$, it is found by the ordinary methods that the following equations must be satisfied for a minimum or a maximum of the integral $I$:

$$
(19)\qquad
\begin{aligned}
f_x H - \frac{\partial(z, q)}{\partial(u, v)} + \frac{\partial(y, r)}{\partial(u, v)} &= 0,\\
f_y H - \frac{\partial(x, r)}{\partial(u, v)} + \frac{\partial(z, p)}{\partial(u, v)} &= 0,\\
f_z H - \frac{\partial(y, p)}{\partial(u, v)} + \frac{\partial(x, q)}{\partial(u, v)} &= 0.
\end{aligned}
$$

These equations are the so-called *Euler-Lagrange equations.*

3. **The figuratrix and its fundamental quantities.** We shall define the *figuratrix* of the integral (6), for the point $P(x, y, z)$ as origin, to be the surface defined by equations (18), in which $p$, $q$, $r$ are rectangular coördinates of a point on the figuratrix, the arguments $x$, $y$, $z$ of $f$ and its derivatives being considered fixed in value. The parameters of this surface are consequently $\tau$ and $\sigma$.

The fundamental quantities of the figuratrix are*

$$
(20)\qquad
\begin{aligned}
E &= (f_\tau \tan\sigma + f_{\tau\sigma})^2/\cos^2\sigma + (f + f_{\sigma\sigma})^2,\\
F &= (f_\tau \tan\sigma + f_{\tau\sigma})\left[(f + f_{\sigma\sigma}) + (f - f_\sigma \tan\sigma + f_{\tau\tau}/\cos^2\sigma)\right],\\
G &= (f_\tau \tan\sigma + f_{\tau\sigma})^2 + \cos^2\sigma(f - f_\sigma \tan\sigma + f_{\tau\tau}/\cos^2\sigma)^2,\\
D &= -\,(f\cos^2\sigma - f_\sigma \sin\sigma\cos\sigma + f_{\tau\tau}),\\
D' &= -\,(f_\tau \tan\sigma + f_{\tau\sigma}),\\
D'' &= -\,(f + f_{\sigma\sigma}),\\
H &= (EG - F^2)^{1/2} = \pm\,(DD'' - D'^2)/\cos\sigma.
\end{aligned}
$$

These fundamental quantities play an important rôle in the theory of minimizing or maximizing the integral (6), as will be shown later.

4. **The transversality condition.** Let us note the relation between the integrands of integrals (6) and (4). We have

$$
(21)\qquad
\begin{aligned}
&f(x,y,z,\tau,\sigma)H\\
&= f\left(x,y,z,\ \arctan\frac{B}{A},\ \arcsin\frac{C}{(A^2 + B^2 + C^2)^{1/2}}\right)(A^2 + B^2 + C^2)^{1/2}\\
&= F(x,y,z,A,B,C).
\end{aligned}
$$

By means of this relation we can transform many of the results of Radon into forms appropriate to the integral (6).

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* See Rider, Transactions.

For example, the *transversality condition*,* which must be satisfied along the boundary of the extremal surface when this boundary is movable along another surface, readily assumes the form

$$p \cos \bar{\tau} \cos \bar{\sigma} + q \sin \bar{\tau} \cos \bar{\sigma} + r \sin \bar{\sigma} = 0, \tag{22}$$

where $p$, $q$, $r$ are the functions (18) and refer to the extremal surface, and $\bar{\tau}$, $\bar{\sigma}$ give the normal direction of the boundary surface.

Let $Q$ be the point $(p, q, r)$ on the figuratrix for the point $P$. Let $PN(\tau, \sigma)$ be the normal to the extremal surface at the point $P$, and $PT(\bar{\tau}, \bar{\sigma})$ the normal to the boundary surface at this point. Then equation (22) states that $PT(\bar{\tau}, \bar{\sigma})$ *must be perpendicular to* $PQ$, *the radius vector of the figuratrix*, $Q$ *being given by the parameter values* $(\tau, \sigma)$.

5. **A geometric interpretation of a certain necessary condition.** Radon† has stated that a certain quadratic form

$$\Phi_{ik} y_i y_k \tag{23}$$

cannot be negative upon an extremal surface if the integral (4) is a minimum, and cannot be positive if the integral is a maximum. In other words, the form (23) must be definite for an extremum. By making use of Radon's equation (11), we find that in the notation of the present paper,

$$\begin{aligned} H^3\Phi_{11} &= -Da_v^2 + 2D'a_vb_v - D''b_v^2, \\ H^3\Phi_{12} = H^3\Phi_{21} &= -Da_ua_v + D'(a_ub_v + a_vb_u) - D''b_ub_v, \\ H^3\Phi_{22} &= -Da_u^2 + 2D'a_ub_u - D''b_u, \end{aligned} \tag{24}$$

where

$$\begin{aligned} a_u &= z_u/\cos^2\sigma, & b_u &= z_u \tan\tau \tan\sigma - y_u/\cos\tau, \\ a_v &= z_v/\cos^2\sigma, & b_v &= z_v \tan\tau \tan\sigma - y_v/\cos\tau. \end{aligned} \tag{25}$$

It may be remarked that the $\Phi_{ik}$ can be expressed in a number of different ways, but for the purpose of establishing the results of this paper the forms given here are sufficient.

If the quadratic form (23) be definite it is necessary that $\Phi_{11}\Phi_{22} - \Phi_{12}^2$ be positive. We readily find that

* Radon, loc. cit., p. 58, equation (16).

† Loc. cit., p. 59.

$$\begin{aligned}\Phi_{11}\Phi_{22} - \Phi_{12}^2 &= (a_u b_v - a_v b_u)^2 (DD'' - D'^2)/H^6 \\ &= \frac{DD'' - D'^2}{H^4 \cos^2 \sigma} = \frac{K}{H^2 \cos^2 \sigma},\end{aligned} \tag{26}$$

$K$ being the total curvature* of the figuratrix. It follows that *the total curvature of the figuratrix must be positive at the point* ($p$, $q$, $r$) *given by the parameter values* ($\tau$, $\sigma$) *which define the direction of the normal to an extremal surface.*

It should be noted that the $\Phi_{ik}$ correspond to $F_1$, $F_2$, $F_3$ in the notation of Kobb† and that Kobb's necessary condition $F_1F_2 - F_3^2 > 0$ is precisely the condition $\Phi_{11}\Phi_{22} - \Phi_{12}^2 > 0$.

6. **The *e*-function.** Let us suppose that we have a surface $\overline{S}$ lying in a field‡ of extremal surfaces. The $e$-function as defined by Radon§ is, in the notation employed in this paper,

$$\begin{aligned}E(x,y,z,A,B,C,\overline{A},\overline{B},\overline{C}) &= F(x,y,z,\overline{A},\overline{B},\overline{C}) - \overline{A}F_A(x,y,z,A,B,C) \\ &\quad - \overline{B}F_B(x,y,z,A,B,C) - \overline{C}F_C(x,y,z,A,B,C),\end{aligned} \tag{27}$$

where $\overline{A}$, $\overline{B}$, $\overline{C}$ define the direction of the normal to the surface $\overline{S}$ at the point ($x$, $y$, $z$), and $A$, $B$, $C$ define the direction of the normal to the extremal surface through that point.

For an integral of the form (6), we find, by making use of the relation (21), that

$$\begin{aligned}E(x,y,z,\tau,\sigma,\bar{\tau},\bar{\sigma}) &= (\overline{A}^2 + \overline{B}^2 + \overline{C}^2)^{1/2}[f(x,y,z,\bar{\tau},\bar{\sigma}) \\ &\quad - (p \cos \bar{\tau} \cos \bar{\sigma} + q \sin \bar{\tau} \cos \bar{\sigma} + r \sin \bar{\sigma})],\end{aligned} \tag{28}$$

where $\bar{\tau}$, $\bar{\sigma}$ refer to the surface $\overline{S}$ and $\tau$, $\sigma$ to the extremal surface. We shall discard the positive factor $(\overline{A}^2 + \overline{B}^2 + \overline{C}^2)^{1/2}$ and define

$$e(\tau,\sigma,\bar{\tau},\bar{\sigma}) = f(\bar{\tau},\bar{\sigma}) - (p \cos \bar{\tau} \cos \bar{\sigma} + q \sin \bar{\tau} \cos \bar{\sigma} + r \sin \bar{\sigma}), \tag{29}$$

omitting, for the sake of simplicity, the arguments $x$, $y$, $z$.

* See Eisenhart, *Differential Geometry*, p. 123.

† Loc. cit.

‡ For the notion of a field of extremal surfaces, see Bolza, loc. cit., pp. 684 ff.

§ Loc. cit., p. 60.

Let now $PQ$ be the radius vector of the figuratrix for the point $P$, and $PS$ the projection of $PQ$ upon the line passing through $P$ in the direction $(\bar{\tau}, \bar{\sigma})$. If $P\overline{N} = f(\bar{\tau}, \bar{\sigma})$ be marked off on the line having the direction $(\bar{\tau}, \bar{\sigma})$, then *the e-function is the line-segment* $SN$. This can be shown precisely as in the author's paper on the figuratrix for simple integrals,* as can also the following statement:

*The e-function is the distance from the point* $\overline{Q}(\bar{\tau}, \bar{\sigma})$ *on the figuratrix to the tangent plane at the point* $Q(\tau, \sigma)$, *except for an infinitesimal of at least the third order with respect to* $(\bar{\tau} - \tau)$ *and* $(\bar{\sigma} - \sigma)$.

* Rider, Transactions.

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